J. Fluid Mech. (1975). *vol.* 72, *part* 1, *pp.* 121–133
Printed in Great Britain

On the eigenvalues of the Orr–Sommerfeld equation

By M. GASTER
Division of Maritime Science, National Physical Laboratory, Teddington, Middlesex, England

AND R. JORDINSON
Mathematics Department, University of Edinburgh, Scotland

(Received 17 April 1975)

The characteristic function defining the eigenvalues of the Orr–Sommerfeld equation is discussed and it is shown how the expected analytic properties of this function can be exploited to generate series expansions defining eigenvalues within the circle of convergence. This technique is applied to the modes arising in the Blasius flat-plate boundary layer (treated as a parallel flow), for which the complex wavenumber α can be expanded as a convergent power series in the complex frequency parameter β in various regions of the β plane. Such power series are effectively equivalent to Fourier expansions and the properties of the latter are used to find the coefficients.

A square-root singularity in the relationship between α and β is found and it is shown how α can, nevertheless, be expressed in terms of β as the sum of one regular series and the square root of a second regular series. The loci of the real and imaginary parts of α have been computed from these series and show the behaviour in the neighbourhood of the branch point.

The series description provides a particularly simple and rapid method of evaluating eigenvalues and their derivatives within any given region.

1. Introduction

We consider small perturbations to a given steady parallel mean flow and discuss some general relationships between the eigenvalues which define modes corresponding to such perturbations. It is shown how this information can be used to derive the coefficients of series expansions defining these eigenvalue relations. The series provide a powerful way of evaluating eigenvalues and their derivatives within the circle of convergence and are particularly convenient when calculation of large numbers of eigenvalues is required. In order to simplify this discussion the following analysis will only be concerned with two-dimensional modes, but the arguments can also be applied to more general problems involving oblique waves.

Small two-dimensional disturbances can conveniently be defined in terms of a perturbation stream function, the properties of which are governed by the linearized and transformed equations of motion. In particular, for mean flows confined within parallel strips of finite width the Fourier transform of the linearized Navier–Stokes equation, evaluated over the frequency (β) and wavenumber (α) plane, provides an ordinary differential equation (the Orr–Sommerfeld equation) for the transform of the stream function ϕ in y. The co-ordinate system for the flow is chosen such that the x axis is parallel to the direction of the mean flow, in which the velocity is assumed to be a function of y only. The stream function is $\psi' = \phi(y, \alpha, \beta) \exp i(\alpha x - \beta t)$, and α and β arise as parameters in the differential equation. The solution of this equation, coupled with the appropriate boundary conditions on ϕ, defines a characteristic equation connecting α and β such that when one quantity is defined the other one appears as an eigenvalue. This characteristic equation is denoted by $F(\alpha, \beta) = 0$. In general a spectrum of eigenvalues is to be expected (Jordinson 1971; Mack 1976), but in most of the flows which have been studied it appears that only the lowest mode is unstable (in some sense) and therefore physically important. Such a mode is defined through a particular zero of the characteristic equation by the values of the parameters α and β. The physical nature of this mode depends on the phases of both α and β; of special interest are those cases with either α or β purely real. Temporal disturbances, which occur when α is real, have complex values of β and grow exponentially in time like $\exp(\beta_i t)$. Such a mode is unstable when $\beta_i > 0$. A spatial mode is described by a real frequency parameter and a complex wavenumber, the imaginary part of which defines the spatial development $\exp(-\alpha_i x)$, indicating a growing unstable wave system when $\alpha_i < 0$ for the usual downstream-propagating disturbance. Apart from these particular modes, which correspond to physically realistic flows, there exists a whole class of disturbances defined by complex values of both α and β. Although such modes may not occur as isolated waves in any physical situation, they are nevertheless useful in Fourier descriptions of more general disturbances which can be considered as composed of a sum or integral of normal modes. Fourier transforms which are valid in the complex plane offer a convenient way of analysing real disturbances through the characteristic equation. For any given mean flow it is in principle possible to obtain this characteristic relationship in numerical terms and hence deduce the flow disturbance. In practice the zero of the characteristic equation associated with the lowest mode is the only one considered, but since the disturbance to the flow is dominated by this mode all important features are represented. Consequently the following discussion concentrates attention solely on the lowest mode.

Numerical calculations of eigenvalues in complex α, β space have been made for various flows and the results are often displayed in the form of a distorted grid of loci of constant α_r and α_i plotted over the complex β plane (or vice versa). These mappings (see Betchov & Criminale 1966; Gaster & Davey 1968; Mattingly & Criminale 1972) show the relationship between the two parameters to be analytic except at isolated points. Since the Orr–Sommerfeld equation, defining the eigenfunction, is regular within the physical flow field in question it follows that all solutions must also be regular, and it can in fact be shown that the characteristic equation $F(\alpha, \beta) = 0$ must be an entire function of both α and β. This results in an analytic relationship between α and β (Gaster 1968) except at isolated branch points where the higher modes are linked. This property has been used to provide a link between spatially and temporally growing waves.

thus enabling the experimental measurements which relate to spatially developing waves to be compared with the temporal disturbances, which are more conveniently considered in theoretical discussions (Gaster 1962). It seems, however, that this analytical property has not been exploited fully in more general discussions of the characteristic equation. Here we show how eigenvalues evaluated around a closed contour can be used to develop a series defining eigenvalues within the contour. The procedure is particularly simple when these regions contain analytic functions, but the branch-point type of singularity, which must arise in some parts of the plane, can be similarly treated without undue complication.

2. Analysis

In the following analysis β is treated as the eigenvalue arising from the solution of the characteristic equation $F(\alpha,\beta) = 0$ when α is prescribed, so that β appears as a function of α. Since $F(\alpha,\beta)$ is in general an entire function of both α and β, it can be expanded about any point (α_0,β_0) as a Taylor series for which the radius of convergence of each of the variables α and β is infinite; the series takes the form

$$\begin{aligned}F(\alpha,\beta) &= F(\alpha_0,\beta_0)+(\alpha-\alpha_0)\frac{\partial F}{\partial \alpha}(\alpha_0,\beta_0)+\tfrac{1}{2}(\alpha-\alpha_0)^2\frac{\partial^2 F}{\partial \alpha^2}(\alpha_0,\beta_0)\\ &\quad+(\beta-\beta_0)\frac{\partial F}{\partial \beta}(\alpha_0,\beta_0)+\tfrac{1}{2}(\beta-\beta_0)^2\frac{\partial^2 F}{\partial \beta^2}(\alpha_0,\beta_0)\\ &\quad+(\alpha-\alpha_0)(\beta-\beta_0)\frac{\partial^2 F}{\partial \alpha\,\partial \beta}(\alpha_0,\beta_0)+\dots\\ &\quad+O[(\alpha-\alpha_0)^3,(\beta-\beta_0)^3,(\alpha-\alpha_0)(\beta-\beta_0)^2,(\alpha-\alpha_0)^2(\beta-\beta_0)]\dots. \qquad (1)\end{aligned}$$

Eigenvalues arise from the zeros of F and thus equating the above series to zero provides an algebraic relation between $\alpha-\alpha_0$ and $\beta-\beta_0$. This equation may be solved by expressing $\beta-\beta_0$ as a power series in $\alpha-\alpha_0$ which has a finite radius of convergence determined by the distance of α from the nearest singularity. In any region not enclosing branch points we can expand about (α_0,β_0), where this point is chosen such that $F(\alpha_0,\beta_0)$ is zero, and obtain

$$\beta-\beta_0 = -(\alpha-\alpha_0)\frac{\partial F}{\partial \alpha}\bigg/\frac{\partial F}{\partial \beta}+\dots+O(\alpha-\alpha_0)^2, \qquad (2)$$

and so find a unique β for every α. It is also possible to obtain the inverse form where α is expressed in terms of β. We note that, in either case, singularities must arise since $\partial F/\partial\alpha$ or $\partial F/\partial\beta$, being themselves entire functions, have at least one zero in the α or β plane respectively (see Gaster 1968). In regions where the Taylor expansion is valid β is an analytic function of α and for such functions it is only necessary to know the value of β round a closed contour in the α plane to define it everywhere inside that region. It is convenient to take the contour to be a circle of radius R in the α plane centred at α_0 so that $\alpha = \alpha_0+R\,e^{i\theta}$.

Since β is an analytic function of α within this circle it must satisfy Laplace's equation and have a general solution of the form

$$\beta = \Sigma C_n r^n e^{in\theta}, \tag{3}$$

where $\alpha - \alpha_0 = re^{i\theta}$, $r \leqslant R$, and the C_n are complex coefficients. Knowing the values of β on the boundary $r = R$ these coefficients can be calculated using the usual Fourier inversion formulae. We are concerned here with the situation where β is specified (i.e. calculated) at a finite number of discrete points on the circle $r = R$. We can express β in the form $\beta(\kappa) = \beta(\theta)\,\delta(\kappa)$, where $\beta(\theta)$ is the function for the continuous behaviour on the circle and $\delta(\kappa)$ is a Dirac comb containing j delta functions equally spaced around the circumference at angular intervals of $2\pi\kappa/j$. The series for the discrete data is given by the convolution of the Fourier series of the continuous function $\beta(\theta)$ and the Dirac comb. This series thus equals that of the continuous function $\beta(\theta)$ only if the coefficients of the latter are zero above the Nyquist folding point. If this condition is not satisfied the coefficients generated by the discrete data set will be aliased by elements above the cut-off. To obtain a useful series representing the function within the circular region it is necessary to use sufficient data points to ensure that the resulting Fourier coefficients become insignificant by the cut-off point. In any specific example we note that the number of terms in the finite Fourier series is equal to half the number of data points around the contour.

3. Practical example

The method outlined in the previous section is applied to the modes which correspond to perturbations of the laminar boundary layer on a flat plate. The mean flow, which is assumed to be parallel, is given by the x component of the Blasius solution, so that the flow field covers the half-plane $y \geqslant 0$, the flat plate lying in the plane $y = 0$. Some of the statements made previously about the behaviour of eigenvalues arising in mean flows confined within parallel strips of finite width may not apply to flows of boundary-layer type. When the flow lies within such a strip the eigenvalues must be discrete since the characteristic function defining them is entire, but this is no longer necessarily so in the case of the boundary layer, where the flow covers a half-plane. In fact it seems likely that the spectrum contains some continuum in addition to any discrete modes which may occur.

In the calculations discussed here we use purely numerical methods to find a set of eigenvalues (round a circular contour, say) from which others may be computed using a power series. The Orr–Sommerfeld equation is discretized in one way or another and appropriate boundary conditions applied at the flat plate and at some finite point in the free stream. The eigenvalues, found either by a matrix iteration technique or by a shooting technique involving direct numerical integration, are basically solutions of a set of linear algebraic equations. The resulting eigenvalues must therefore be discrete although in the present example there may well be a continuous spectrum associated with the differential equation as well. The present discussion, however, is concerned

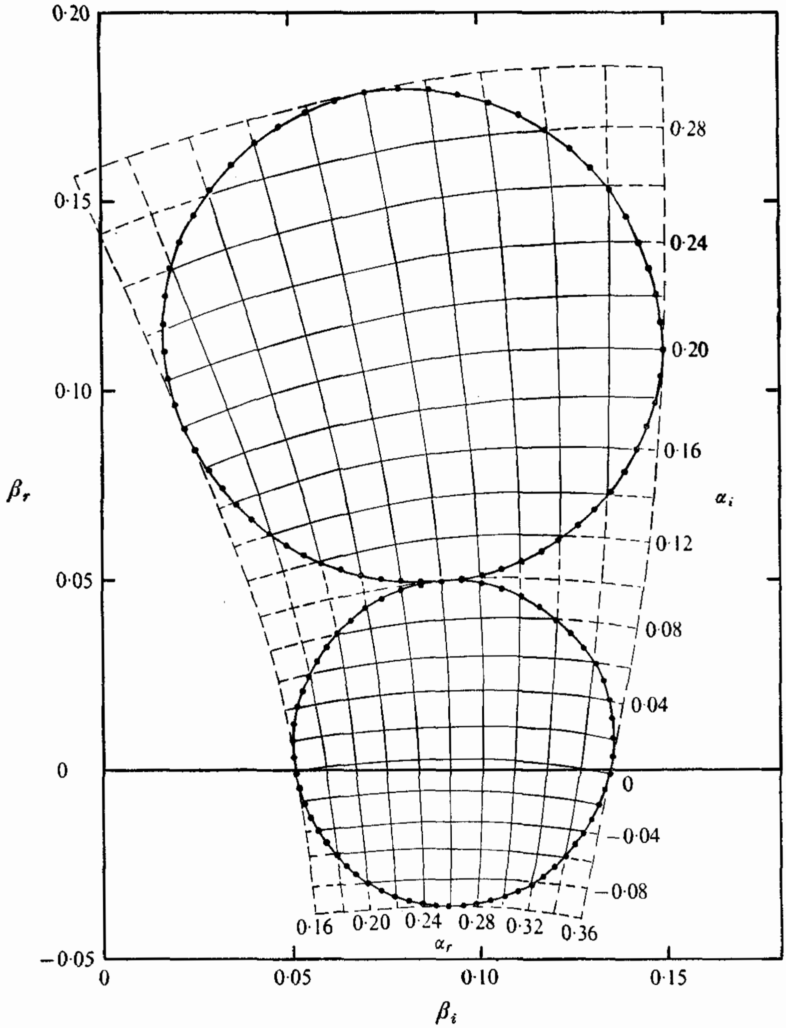

FIGURE 1. Loci of constant α_r and α_i on the β plane for two circular regions in the α plane of radius 0·1 and centres (0·26, 0) and (0·26, 0·1).

with the lowest modes and these are in any case known to be discrete for the boundary-layer problem. The following treatment must nevertheless be applicable to *all* the discrete eigenvalues generated numerically since these exhibit the analytic behaviour expected for the discrete modes of a typical strip problem.

In this example β eigenvalues were calculated from the Orr–Sommerfeld equation for 60 values of α equally spaced around the circle $|\alpha - \alpha_0| = 0{\cdot}1$, where $\alpha_0 = (0{\cdot}26, 0)$. The Reynolds number, based on the displacement thickness, was taken to be 1000. The eigenvalues were found using a matrix method with 80 intervals across the boundary layer (Jordinson 1970) and are plotted on the β plane in figure 1. They lie unequally spaced on an almost circular closed contour. The coefficients of the Fourier series defined by these points were computed and the first 18 are shown in table 1. The coefficients decrease quite steadily with increasing n and reach a lower limit roughly equal to the rounding

n	A_n	B_n
0	9·17, −2	3·11, −2
1	4·23, −2	5·90, −4
2	9·83, −4	−3·87, −3
3	−4·74, −4	−5·87, −4
4	−5·99, −5	5·65, −5
5	−7·68, −6	−4·45, −5
6	−8·91, −6	9·97, −6
7	−1·30, −6	−4·79, −6
8	−3·38, −7	1·94, −6
9	−4·95, −7	−6·65, −7
10	1·73, −7	3·08, −7
11	−1·50, −7	−6·98, −8
12	6·74, −8	2·63, −8
13	−3·54, −8	3·91, −9
14	1·51, −8	−4·82, −9
15	−6·32, −9	4·87, −9
16	1·95, −9	−3·23, −9
17	−7·12, −10	1·47, −9
18	−2·11, −10	−1·16, −9

TABLE 1. Fourier coefficients based on 60 points around the circle in the α plane centred at (0·26, 0) with a radius of 0·1. The coefficients A_n and B_n are the real and imaginary parts of R^nC_n, where R is 0·1.

error of the computer used in the evaluation of the β's around the circle. A better impression of the behaviour of the coefficients is provided by the logarithmic plot of the modulus $A_n^2+B_n^2$ *vs.* n shown on figure 2. Since the eigenvalues themselves are only accurate to 5 or 6 decimal places, nothing is gained by using more than about 12 terms of the series to represent β within the prescribed region. This implies that β is a smooth enough function in this particular example to be adequately represented by 24 points on the circumference of the circle in the α plane. Values of β for different α's have been calculated from the full series and from the series truncated at the twelfth term, and these are compared in table 2 with the eigenvalues generated directly from the Orr–Sommerfeld equation by the matrix routine. The agreement achieved is very satisfactory within the contour region both for the full and the truncated series. Significant differences do not arise, even for the latter series, until the extremities of the region are reached.

The series was also used to calculate β_r and β_i for an array of points covering the square circumscribing the circle in the α plane. This information is displayed on figure 1 in the form of a grid of lines of constant α_r and α_i on the β plane. Similar calculations were made for an adjoining region and the grid lines have been continued into this area of the β plane.

4. Treatment of a singular point

Attempts to map α_r and α_i loci over certain regions of the β plane by the usual process of plotting directly evaluated eigenvalues produced a confused picture. This arose because higher modes with similar values of β existed in

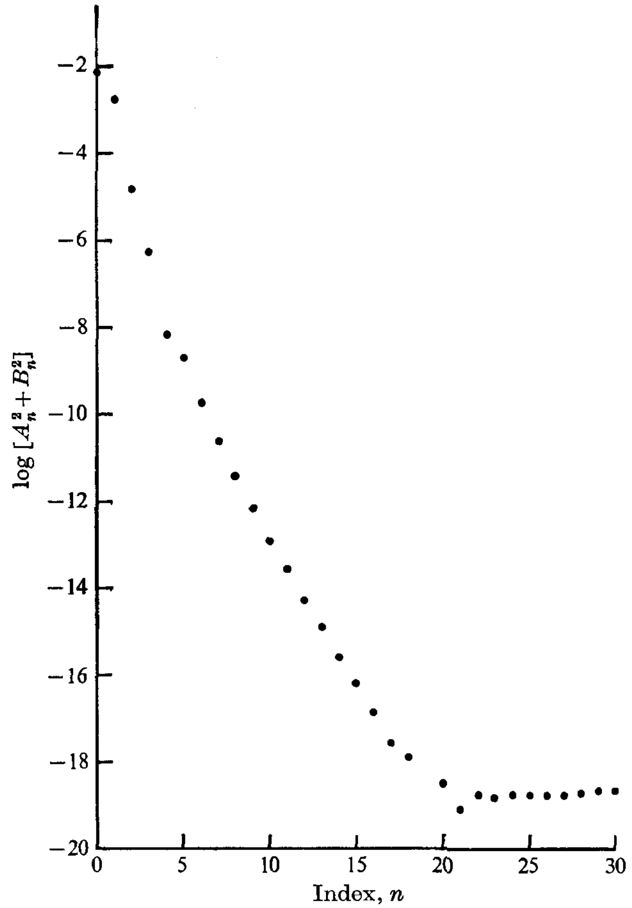

FIGURE 2. Magnitudes of the Fourier coefficients generated from the circle of centre (0·26, 0) and radius 0·1 in the α plane. A_n and B_n are the real and imaginary parts of R^nC_n of equation (3).

		β_r	β_i
$\alpha_r = 0{\cdot}26$, $\alpha_i = 0$	(*a*)	0·09165793	0·00310804
	(*b*)	0·09165793	0·00310804
	(*c*)	0·09165793	0·00310804
$\alpha_r = 0{\cdot}2954$, $\alpha_i = 0{\cdot}0354$	(*a*)	0·10745666	0·01851389
	(*b*)	0·10745669	0·01851391
	(*c*)	0·10745669	0·01851391
$\alpha_r = 0{\cdot}3307$, $\alpha_i = 0{\cdot}0707$	(*a*)	0·12578484	0·03446345
	(*b*)	0·12578484	0·03446345
	(*c*)	0·12578482	0·03446343
$\alpha_r = 0{\cdot}36$, $\alpha_i = 0{\cdot}10$	(*a*)	0·14330725	0·04819733
	(*b*)	0·14330714	0·04819702
	(*c*)	0·14330521	0·04819636

TABLE 2. Values of β calculated from (*a*) direct solution of the Orr–Sommerfeld equation, (*b*) full Fourier series and (*c*) Fourier series with 12 terms.

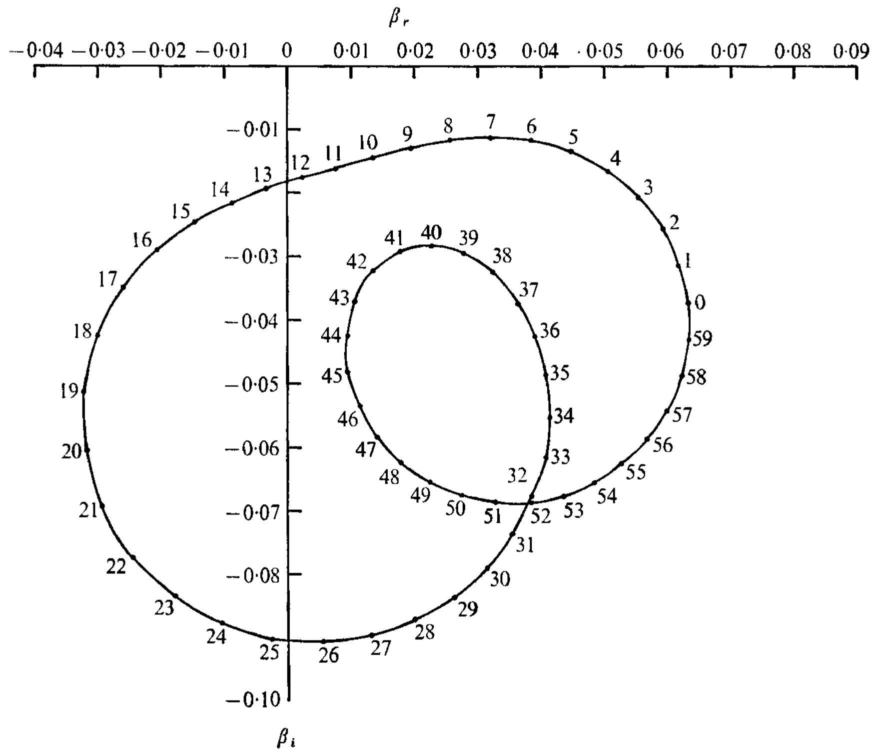

FIGURE 3. Loci of points around the circumference of the circle of radius 0·08 centred at (0·1, −0·1) on the α plane. Points 1–29 refer to the first circuit. Points 30–59 refer to the second circuit.

these regions. The pattern of behavour indicated the presence of a square-root branch-point singularity in the function $\beta(\alpha)$ similar to those found in other cases (Betchov & Criminale 1966; Gaster 1968). A procedure similar to that described in the previous section was adopted, although modified to account for the presence of the singularity. The resulting expansions were used to locate the singularity and obtain the complete solution in its neighbourhood.

4.1. *Analysis*

The values of β for 30 equally spaced steps round a circle in the α plane of radius $R = 0{\cdot}08$ and centre $\alpha_0(0{\cdot}1, -0{\cdot}1)$ are plotted in figure 3. It was found that one circuit in the α plane did not map into a closed loop in the β plane and a second circuit was needed in the α plane to complete the loop in the β plane. β is clearly double valued inside this region and consequently a different series representation is needed. We consider briefly this aspect of the problem.

Suppose that the singularity is at (α_1, β_1), where $\beta_1 = \beta(\alpha_1)$; the expansion of α in terms of β in the neighbourhood of the singularity is of the form

$$\alpha - \alpha_1 = a_2(\beta-\beta_1)^2 + a_3(\beta-\beta_1)^3 + \ldots,$$

which implies that the disturbance has infinite group velocity at α_1. It follows that the inverse series for β in terms of α in the corresponding neighbourhood is of the form

$$\begin{aligned}\beta-\beta_1 &= b_1(\alpha-\alpha_1)^{\frac{1}{2}}+b_2(\alpha-\alpha_1)+b_3(\alpha-\alpha_1)^{\frac{3}{2}}+\dots\\ &= (\alpha-\alpha_1)^{\frac{1}{2}}\times \text{a regular series in powers of } \alpha-\alpha_1\\ &\quad+\text{another regular series in powers of } \alpha-\alpha_1+\dots, \qquad (4)\end{aligned}$$

where $|\alpha-\alpha_0| < d$, say, and d is the distance from α_1 to the next singularity. In the current problem the basic circle is centred at α_0 and the location of α_1 is unknown. It is possible to write down an expansion similar to (4) of the form

$$\begin{aligned}\beta-\beta_0 &= (\alpha-\alpha_0)^{\frac{1}{2}}\times \text{a regular series in powers of } \alpha-\alpha_0\\ &\quad+\text{another regular series in powers of } \alpha-\alpha_0+\dots, \qquad (5)\end{aligned}$$

where $\beta_0=\beta(\alpha_0)$, subject to the conditions (i) that $|\alpha-\alpha_0| > |\alpha_0-\alpha_1|$ and (ii) that no other singularity lies within the circle $|\alpha-\alpha_0| = R$. So, as the first circuit of this circle is made, the values of β will depend on the sign of the first term in (5); this sign changes when the second circuit is commenced and gives rise to a second branch of β. This result may be expressed more concisely in the form

$$\beta = X \pm Y^{\frac{1}{2}},$$

where X is the regular component and $Y^{\frac{1}{2}}$ corresponds to the first series on the right-hand side of (5).

4.2. *Numerical details*

Let us define the values of β on the first circuit of the α-plane circle as $\beta(q)$, where q goes from 0 to 29, so that $\beta(q+30)$ denotes values on the second circuit. It follows that $\frac{1}{2}[\beta(q)+\beta(q+30)]$ must be related to the regular component X and corresponding to this we define a sequence $\beta_1(q)$ by

$$\beta_1(q) = \tfrac{1}{2}[\beta(q)+\beta(q+30)],$$

where q goes from 0 to 29. Since $\beta(q)$ and $\beta(q+60)$ are identical it is clear that $\beta_1(q)$ has the same value as $\beta_1(q+30)$, and $\beta_1(q)$ is therefore a single-valued function of α. The singular part, corresponding to $Y^{\frac{1}{2}}$, is given by the differences in the values of β on the two circuits, so we define a second sequence $\beta_2(q)$ by

$$\beta_2(q) = \tfrac{1}{2}[\beta(q)-\beta(q+30)],$$

where q goes from 0 to 29 as before. This quantity changes sign on alternate circuits, so that

$$\beta_2(q) = -\beta_2(q+30).$$

The regular component X can be expressed directly as a Fourier series generated from the values of $\beta_1(q)$. The singular component $Y^{\frac{1}{2}}$ becomes regular when squared, so that Y can be expressed as a Fourier series from values of $[\beta_2(q)]^2$. Thus we can write β in the form

$$\beta = \sum_{n=0}^{m} X_n r^n e^{in\theta} \pm \left(\sum_{n=0}^{m} Y_n r^n e^{in\theta}\right)^{\frac{1}{2}},$$

where $\alpha-\alpha_0 = re^{i\theta}$ and the complex coefficients X_n and Y_n are found using the method already described from the values of $\beta_1(q)$ and $\beta_2(q)$ round the circuit.

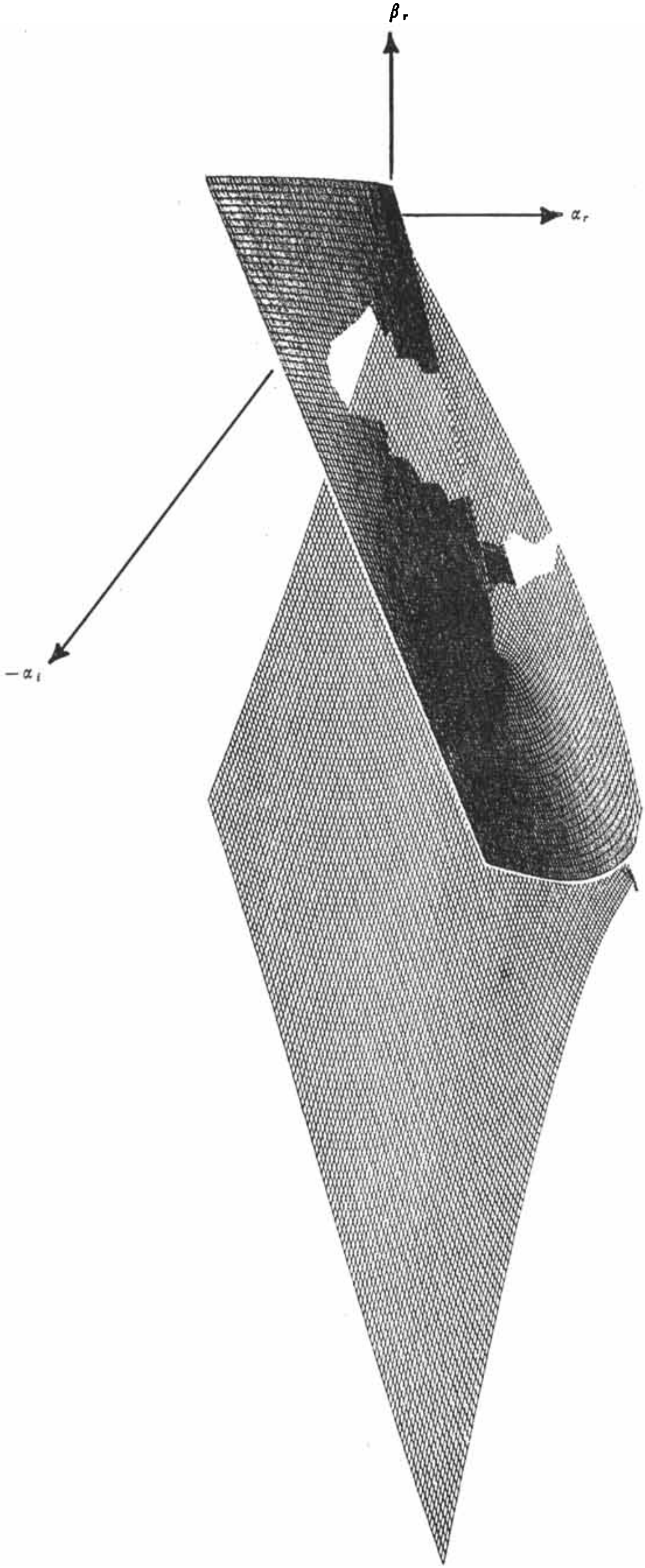

FIGURE 4. Riemann surface formed by the display of β_r on the α plane near the singular point.

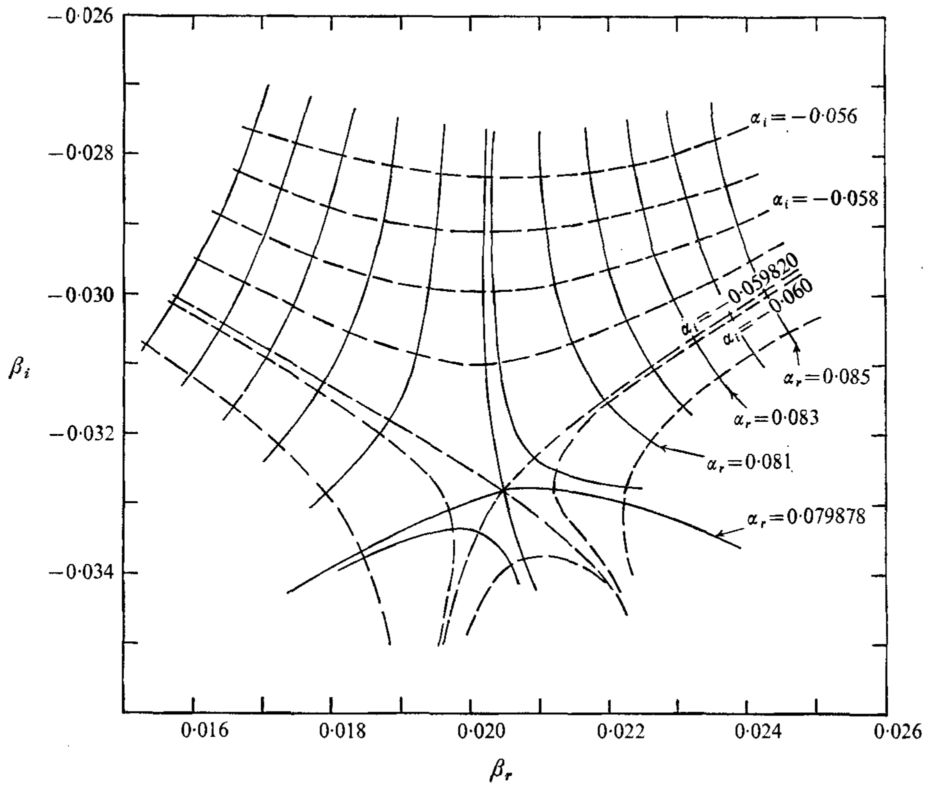

FIGURE 5. Loci of constant α_r and α_i plotted on the β plane near the singularity.

The two values of β are then easily calculated from these series at any point inside the enclosed circular region in the α plane.

In order to display the behaviour of the function $\beta(\alpha)$ values of β were calculated for an array of points on the α plane. The real part β_r is shown as the ordinate in a perspective projection on the α plane. For each of 101×101 grid-point values of α there are two values of β_r, and the two sheets have been combined in this display in an attempt to show the Riemann surfaces. This picture (figure 4) shows clearly the square-root singularity, in the neighbourhood of which the gradients of β_r become steep and multi-valued. The position of this singular point can easily be found from the series by an iterative scheme. The values are

$$\alpha_1 = 0{\cdot}079878 - 0{\cdot}059820i, \quad \beta_1 = 0{\cdot}020488 - 0{\cdot}032855i.$$

The behaviour of α near this singular point is shown in figure 5.

The power of the series technique is well demonstrated by this example, where the computations involved some 20000 eigenvalues. The time taken to evaluate these from the series was about the same as that required for the computation of, say, half-a-dozen eigenvalues by direct solution of the Orr–Sommerfeld equation.

The region around the singular point is shown in detail by the plot of lines of constant α_r and α_i on the β plane. This is just the type of picture previously shown by Betchov & Criminale for other types of flow.

	Re $\int \beta \, d\alpha$	Im $\int \beta \, d\alpha$
1st circuit	−0·3581125168	−0·19878723528
2nd circuit	+0·3581125168	+0·19878723528

TABLE 3

5. Analytical nature of the eigenvalues and line integrals

The analytical nature of the eigenvalues obtained by purely numerical techniques can be demonstrated by evaluating

$$\int_C \beta \, d\alpha,$$

where C is, say, the circle $|\alpha - \alpha_0| = R$ defined in the previous section. If C encloses a singularity the line integral round one circuit will not be zero and a further circuit will be necessary to return the value to zero. Table 3 below gives some typical results for the circle with α_0 and R as defined earlier.

The eigenvalues themselves are correct to only 5 or 6 decimal places so it is evident that the eigenvalues of the discretized differential equation have analytic properties in their own right irrespective of whether or not they are close approximations to the eigenvalues of the original differential equation. This result is consistent with the earlier discussion on the nature of discrete eigenvalues.

6. Discussion and concluding remarks

The representation of β in terms of α by a series enables rapid calculation of eigenvalues within a given region. This is particularly useful when large numbers of data points are needed to display the function either as a grid of constant β_r and β_r loci on the α plane or as a perspective projection. The character of the eigenvalue behaviour can readily be identified and any branch point found.

The series description of the eigenvalue relation simplifies any calculations involving these parameters and their derivatives, which in turn can also be expressed directly as a series. A particular example where this method has been used to good effect arises in the calculation of disturbances composed of large numbers of modes. Wave packets, which can be defined in terms of integrals of isolated modes, can be evaluated from asymptotic expansions involving the functions $\beta(\alpha)$, $\partial\beta(\alpha)/\partial\alpha$ and $\partial^2\beta(\alpha)/\partial\alpha^2$ (Gaster & Davey 1968). A direct solution of this problem using the Orr–Sommerfeld equation to provide the quantities $\beta(\alpha)$ etc. proved to be tedious and very costly in terms of computing time. The present approach, where $\beta(\alpha)$ is obtained as a convergent series, provides a simpler and much faster way of evaluating the form of the disturbance in the physical plane.

Contour integration of eigenvalues round a circle has demonstrated the remarkable analytic properties of the modes of the discretized differential equation. In practice it appears that such modes follow a well-behaved pattern. They are isolated and arise from the zeros of a characteristic function which is entire in α and β. Each of the zeros represents one mode and it seems, therefore, that the higher modes which have been found by Jordinson (1971), Mack (1976) and others must be linked through the branch points of the function described earlier. In the numerical example of §4.2 two close values of β were generated for every α; one value related to the usual unstable lowest mode of the system, whilst the other related to a damped higher mode. The spectrum of modes for the boundary layer has been investigated by Mack for a specific real wave-number of 0·308 and a Reynolds number of 1000 in our notation. The present calculations were extended along the real axis and it was found that the values of β_r and β_i for $\alpha = 0{\cdot}308$ were virtually identical to Mack's second mode. There can be no doubt that the other higher modes are also coupled through further branch points and a more direct way of finding them might well be through some contour integration procedure of the type discussed in this paper.

REFERENCES

Betchov, R. & Criminale, W. O. 1966 *Phys. Fluids*, **9**, 359.

Gaster, M. 1962 *J. Fluid Mech.* **14**, 222.

Gaster, M. 1968 *Phys. Fluids*, **11**, 723.

Gaster, M. & Davey, A. 1968 *J. Fluid Mech.* **32**, 801.

Jordinson, R. 1970 *J. Fluid Mech.* **43**, 801.

Jordinson, R. 1971 *Phys. Fluids*, **14**, 2535.

Mack, L. M. 1976 *J. Fluid Mech.* To appear.

Mattingly, G. E. & Criminale, W. O. 1972 *J. Fluid Mech.* **51**, 233.